PERMUTATION STRINGS

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Problem: What is the length of the shortest string on an alphabet of n symbols that contains all $n!$ permutations of the alphabet as (contiguous) substrings?

Partial Solution: Minimum length strings for $n = 1, 2, 3$ are, respectively, 1, 121, and 123121321. Their lengths are $1 = 1$, $3 = 1!+2!$, and $9 = 1!+2!+3!$. Given any string that contains all permutations of an alphabet of $n-1$ symbols, a new string containing all permutations of n symbols can be generated by the following procedure: replace each of the $(n-1)!$ permutations of the form $j_1 \ldots j_{n-1}$ with the string $j_1 \ldots j_{n-1} n j_1 \ldots j_{n-1}$. Overlaps among permutations in the old string are maintained in the replacements and each of the $(n-1)!$ replaced strings grows in length by n symbols. Therefore, the new string is $n! = n \times (n-1)!$ symbols longer than the old string. The following depicts the construction as discussed for transforming the upper-bound solution from $n = 3$ to $n = 4$. Note, the output substrings, e.g., 1234123, contain all cycles of their first four characters.

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123               1234123
 231                   2314231
  312                       3124312
    213                          2134213
     132                             1324132
      321                                3214321
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123121321   =>    123412314231243121342132413214321
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A simple induction argument establishes the fact that $U(n) = \sum_{i=1}^{n} i!$ is an upper bound on the necessary string length. Further, $n!$ is clearly a lower bound. Note, $(U(n) - n!)/n! \approx 1/n$ so the order of magnitude is correct. I conjecture, but cannot prove that $U(n)$ is the solution to the stated problem.[2]

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[2] Since writing the above, Robin Houston, *Tackling the Minimal Superpermutation Problem*, arXiv:1408.5108v1 [math.CO] 21 Aug 2014, has been brought to my attention. Houston produces a counterexample to my conjecture for $n = 6$. Thus, $\sum_{i=1}^{n} i!$ is just an upper bound for $U(n)$.